Space-charge-dominated bunched beams in the frequency domain

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The effect of space charge on high-phase-space-density bunched beams in an ion storage ring with electron cooling is investigated in the frequency domain for bunches in the vicinity of the local microwave instability threshold. Space charge causes a splitting of synchrotron satellites into incoherent and coherent lines. Results of measurements of the splitting and shift are compared with analytical formulas as well as with simulation of the noise spectra by multiparticle computer simulation. Coherent frequencies of the quadrupole and sextupole modes of Gaussian bunches are found to deviate slightly from the analytical mode frequencies. Consequences for the interaction with narrow-band impedances and on the possible accuracy of $\Delta p/p$ momentum-spread measurements for high-phase-space-density bunches are briefly discussed.

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I. INTRODUCTION

High-phase-space-density bunched beams dominated by space charge can be achieved in high-intensity proton synchrotrons or in storage rings with electron cooling. This condition leads to the well-known "potential well flattening" effect due to space charge below the transition energy. Above the transition space charge increases the applied rf voltage as was found experimentally in the intersecting storage rings for protons of about 20 GeV energy [1]. There, contrary to the present study of cooler rings, the inductive-wall impedance was found to be dominant over the space-charge impedance and explained as the origin of bunch lengthening due to self-forces that reduce the effective rf voltage. In Ref. [1] it was suggested that these self-forces can be determined indirectly—as a direct measurement of the incoherent effective synchrotron frequency is difficult—by measuring the shift in the coherent quadrupole or sextupole mode frequency and using an analytical linear relationship between the coherent and incoherent shifts.

The space-charge potential well flattening for nonrelativistic energies is particularly important if very high phase-space densities are considered. This can be the case, for example, in cooler storage rings or in storage rings considered for heavy ion inertial fusion [2]. The flattening is of practical importance since the effective synchrotron frequency ω_s enters into the expression for the momentum spread

$$\frac{\Delta p}{p} = \pm \frac{\omega_s z_0}{|\eta|\beta c}, \tag{1}$$

with z_0 the bunch (half) length and η the slip factor in circular machines. For negligible space charge ($\omega_s = \omega_{s0}$) the momentum spread is determined by measuring z_0; for increasing phase-space density z_0 approaches a limiting value given by intensity and we obtain $\Delta p/p \propto \omega_s$. This behavior was recently discussed in connection with measurements of cooled proton bunches, where the attainment of space-charge-dominated bunches with practically vanishing momentum spread was concluded from the measurements [3,4].

In the context of bunch instabilities the space-charge-induced splitting and shift of synchrotron satellites need to be considered if very narrow band impedances exist that overlap with individual lines of the bunch spectrum. It is not obvious that the coherent eigenfrequencies derived analytically for strictly parabolic bunches also apply to Gaussian bunch shapes and strong potential-well flattening. In the present paper we therefore investigate this subject by direct measurements of both the incoherent and coherent frequencies and a comparison with the noise spectra derived from computer simulation of different bunch models ("simulation noise").

For an interpretation of both experimental and simulation data we briefly summarize results from analytical theory. The bunch equilibrium, which allows one to derive exact expressions for coherent frequencies, is that of a parabolic bunch with line density $\lambda(z,t) = \lambda_0\left(1 - z^2/z_0^2\right)$, which is consistent with the elliptic phase-space distribution $f(z,z') \propto [1 - z^2/z_0^2 - z'^2/(\eta\Delta p/p)^2]^{1/2}$, where $z' \equiv dz/ds$ ($s = \beta ct$) [5,6]. This is based on the asumption that the externally applied rf force varies linearly from the bunch center and that the bunch is long compared with the beam pipe diameter. These conditions are responsible for the longitudinal space-charge electric field to vary also linearly with longitudinal distance from the bunch center: $E_z = (-qeg/4\pi\epsilon_0)(\partial\lambda/\partial z)$. Here q is the charge state of the ion and g a geometry factor given by $g = 0.5 + 2\ln(R_p/R_b)$, which is valid for a circular pipe and includes an averaging over the transverse beam dimension [7]. In this expression, R_p is the pipe and R_b the beam radius, which is assumed to be constant along the bunch.

The effective (harmonic) potential $V_{\rm eff}$ is the sum of the focusing rf potential V_0 and the defocusing space-charge potential $V_{\rm SC}$. Hence the zero-space-charge synchrotron frequency ω_{s0} is given by

$$\omega_{s0}^2 = \frac{qeh|\eta|V_0}{2\pi R^2 A m_p \gamma} , \tag{2}$$

with h the rf harmonic, A the ion mass, and R the ring radius. This leads to an effective incoherent synchrotron frequency ω_s in the flattened potential according to

$$\omega_s^2 = \omega_{s0}^2 - \frac{3q^2 r_p c^2 g|\eta|N}{2A\gamma^3 z_0^3} , \tag{3}$$

with r_p the classical proton radius and N the number of ions per bunch. Eigenfrequencies of coherent oscillations of the stationary bunch have been obtained for different mode numbers m in Ref. [8] by means of Vlasov's equation. In the bunch frame the resulting frequencies can be written in terms of the incoherent and zero-space-charge synchrotron frequency

$$\omega_1^2 = \omega_{s0}^2, \tag{4}$$

$$\omega_2^2 = 3\omega_{s0}^2 + \omega_s^2, \tag{5}$$

$$\omega_3^2 = 3\omega_{s0}^2 + 2\omega_s^2 + (9\omega_{s0}^4 + 3\omega_{s0}^2\omega_s^2 + 4\omega_s^4)^{1/2}, \tag{6}$$

$$\omega_4^2 = 5\omega_{s0}^2 + 5\omega_s^2 + (25\omega_{s0}^4 - 5\omega_{s0}^2\omega_s^2 + 16\omega_s^4)^{1/2}. \tag{7}$$

The mode $m = 1$ describes a "dipole" mode or rigid bunch oscillation, $m = 2$ a "quadrupole" or length oscillation, $m = 3$ a "sextupole" mode with the bunch getting asymmetric, and $m = 4$ an "octupole" mode (see Fig. 1). For zero space charge these coherent frequencies are multiples of the synchrotron frequency. Note that the dipole mode frequency is unchanged by space charge since it has no effect on the motion of the bunch center. For small space-charge shift one finds from Eq. (5) a linear relationship

$$\omega_2 - 2\omega_{s0} = \frac{1}{2}(\omega_s - \omega_{s0}), \tag{8}$$

which was suggested in Ref. [1] as an indirect way to determine the incoherent ω_s and thus the strength of self-forces.

II. MEASURED COHERENT AND INCOHERENT NOISE SPECTRA

The existence of coherent eigenfrequencies has important consequences on the measured noise spectra. For a single bunch the Fourier transform of the signal on a longitudinal pickup contains "coherent" lines at multiples of the revolution frequency due to the periodic passage of the bunch. Assuming a bunch that is perfectly matched to the rf bucket, Schottky noise in its proper sense is the noise from the synchrotron oscillations with random phases of a finite number of particles. The random phases give rise to statistical fluctuations of the electrical current and induces a voltage on the pickup, which is Fourier transformed (for a mathematical treatment see Ref. [9]).

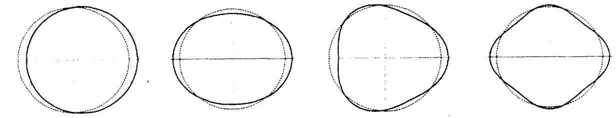

FIG. 1. Schematic contours of bunch modes $m = 1$ (dipole), $m = 2$ (quadrupole), $m = 3$ (sextupole), and $m = 4$ (octupole) in synchrotron phase space.

This procedure results in the "incoherent" Schottky spectrum consisting of synchrotron satellites at multiples of ω_s.

For high intensity there is also coherent noise due to phase correlations associated with the above-described coherent oscillations. These oscillations produce satellites distinct from the multiples of the single-particle synchrotron frequency ω_s. Their strength depends on the competition of excitation and damping. It should be noted that for ion bunches oscillating in a strictly harmonic potential there is no phase mixing or Landau damping as all particles have the same synchrotron frequency. A length oscillation induced by a small jump in the rf amplitude, for example, may thus persist for a long time and can be observed as a coherent signal. We note that due to the absence of damping the notion of coherent noise of a bunched beam is applicable to low intensity as well. The coherent signals, in particular, can be excited artificially by appropriately perturbing the rf bucket (rf phase or amplitude jump or higher harmonic rf cavity).

We have measured, in the ESR storage ring for heavy ions, bunch profiles by using the position pickups and the corresponding fast Fourier transform spectra of a Ne^{10+} beam of 244 MeV/u and a rf voltage of 100 V (harmonic 2) in the presence of electron cooling. In the example of Fig. 2 we had 1.5×10^8 particles (circulating beam current $I = 840\ \mu A$). The spectrum was measured at the tenth harmonic of the revolution frequency (1.69 MHz). Typically 64 spectra have been averaged in order to raise the confidence level for the incoherent noise. Such measurements have been successful only for sufficiently stable operation of the electron cooler and relatively low intensity. For intensities exceeding 1–2 mA we have found unstable behavior of the bunches, which made the measurements impossible.

The spectrum contains a central line at a multiple of the revolution frequency and a large number of sidebands. Some of the sidebands are at multiples of 50 Hz and not related to bunch frequencies. We assume they are caused by the power supply of the cooler, rf system, or magnets. The first relevant satellite is a small peak shifted 185 Hz from the central line; a much stronger peak at 230 Hz is ascribed to coherent dipole motion (m=1) as it occurs exactly at the same frequency for different intensities. Hence we associate the small peak with the effective (incoherent) synchrotron frequency ω_s and the larger peak with the zero-space-charge synchrotron frequency ω_{s0}. The splitting into these two lines is obviously due to space charge. The applied rf voltage of 100 V is thus flattened to an effective voltage of 65 V. It is noted

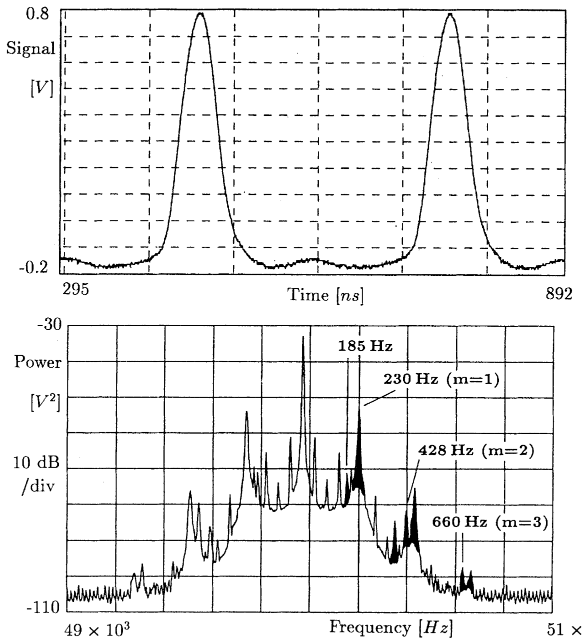

FIG. 2. Top, bunch signal of electron cooled bunches (second harmonic); bottom, corresponding FFT power spectrum (64 averages) at the tenth harmonic with space-charge splitting of synchrotron satellites (relevant peaks filled black).

that the measurement also shows the second harmonic of these frequencies at 370 and 460 Hz, furthermore lines at 428 Hz as well as at 660 and 690 Hz. This coordination of lines and frequencies is also supported by the findings from computer simulation noise (see Sec. III). Using ω_s and ω_{s0} we calculate from Eqs. (5) and (6) the frequency for the quadrupole mode ($m = 2$), as 439 Hz and for the sextupole mode ($m = 3$) as 644 Hz, which can be identified with the measured lines at 428 and 660 Hz. Hence space charge gives rise to a splitting of the $2\omega_{s0}$ satellite into three lines: the incoherent line $2\omega_s$, the coherent quadrupole line ($m = 2$), and the second harmonic of the coherent dipole line ($m = 1$). We emphasize that in the bunch frame the mth mode of Fig. 1 is related to the mth sideband; in the laboratory frame it also generates higher harmonic lines at multiples of the fundamental frequency, hence it is not quite correct to identify, as frequently done in the literature, the mth sideband with the mth-order mode.

By using Eq. (3) and a measured bunch length of 80 ns (full length of equivalent parabolic bunch) we can derive from the frequency measurements the g factor of this example as 7. This allows us to estimate $R_p/R_b \approx 30$, which is an averaged ratio around the machine. With an averaged pipe radius of $R_p \approx 0.06$ m, this is consistent with an (unnormalized) emittance of 1π mm mrad. Such a value is in good agreement with emittance measurements of coasting beams of the same local current by a charge exchange detector [11].

The agreement between analytical and measured quadrupole and sextupole mode frequencies is satisfactory, if one keeps in mind that the analytical formulas are based on the parabolic bunch model, whereas the real bunches are more likely to be Gaussian [12]. Measurements at different intensities indicate, however, that the quadrupole frequencies are systematically slightly below and the sextupole frequencies above the analytical values. A closer comparison is the subject of Sec. IV.

III. COMPUTER SIMULATION OF NOISE

We have used the particle-in-cell computer code SCOP-RZ, which assumes rotational symmetry around the beam axis. Particles are traced by solving Newton's equation of motion for typically 50 000 simulation particles. Each time step we create a density on a mesh in r,z and solve Poisson's equation by assuming an infinitely conducting cylindrical pipe. Space charge is thus self-consistently taken into account, whereas other sources of impedance have been ignored in this context.

As a check on the accuracy of space-charge calculation in the simulation program we have evaluated the self-consistently calculated electric field induced by a parabolic bunch with uniform radial density profile (independent of z) for different ratios R_p/R_b and assuming a long bunch with $z_0/R_p = 150$. The self-electric-field on axis is shown in Fig. 3 for $R_p/R_b = 8$. It is seen that the electric field has a linear dependence on z, which is a necessary condition for the validity of the elliptic phase-space distribution. We note that this "cylindrical" bunch model differs from an ellipsoidal bunch (R_b dependent on z), where E_z is found to be a nonlinear function of z in the long-bunch limit [10]. We find that the g factor determined from the electric field agrees with the analytical formula within 1% deviation. This g factor is identical to that of longitudinal space-charge waves on coasting beams, where the transverse density remains invariant (see, for example, Ref. [7]).

In analogy to the real experiment we evaluate the noise of computer simulation by recording the line density over a large number of time steps and carrying out a Fourier transformation. As in real beams the simulation noise is caused by the discreteness of particles. The resulting fluctuations of E_z depend on the number of simulation

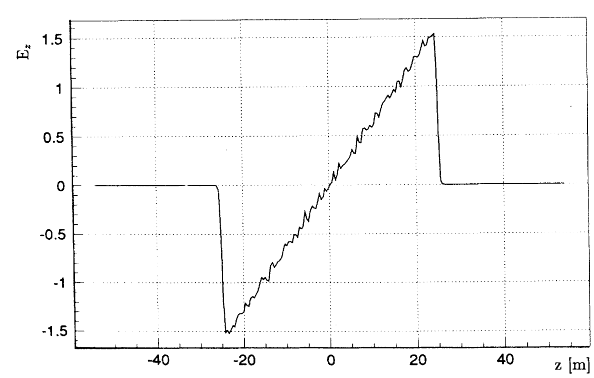

FIG. 3. Self-consistent E_z (arbitrary units) in the computer simulation of a bunch with a parabolic current profile and cylindrical spatial boundary.

particles. For the present example they are shown in Fig. 3. For the same reasons as in the experiment we also carry out an averaging of spectra obtained in subsequent time intervals. Simulation results for zero space charge and with finite space charge are shown in Fig. 4 for a parabolic bunch with elliptic phase-space distribution in a harmonic rf potential. The synchrotron frequency has been increased in the simulation in order to reduce computing time. The identification of mode frequencies is made using the analytical formulas of Eqs. (4)–(7), which apply to the same phase-space distribution. It should be noted that, in contrast to the analytical frequency calculations, the simulation noise is seen in a frame of reference where the bunch is moving, hence multiples of the bunch oscillation frequencies are seen as well. The incoherent (single-particle) synchrotron frequency is the lowest observable frequency followed by the dipole frequency. The second satellite splits into the second harmonics of the incoherent frequency and the dipole frequency as well as the quadrupole mode frequency. The agreement with the analytical formulas is found to be excellent, as will be summarized in Fig. 5 for different strengths of space charge.

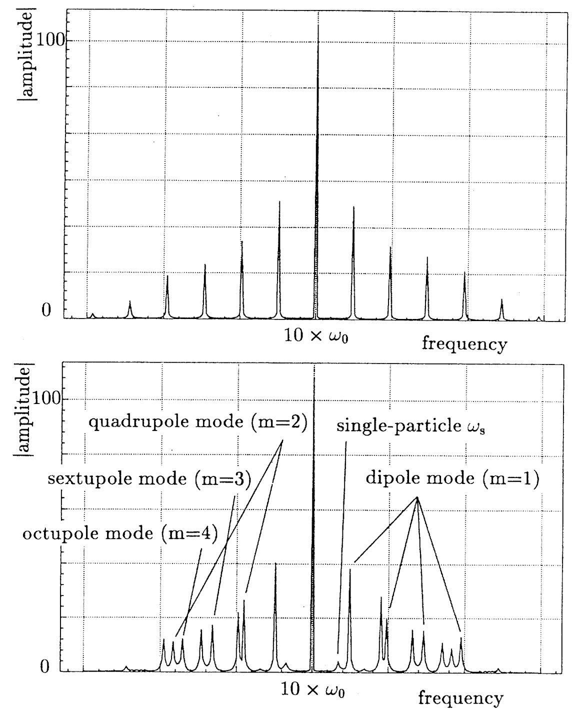

FIG. 4. Top, synchrotron satellites from computer "simulation noise" without space charge for the parabolic bunch; bottom, space-charge splitting into incoherent and coherent lines (frequencies and amplitudes in arbitrary units).

IV. COMPARISON AND DISCUSSION OF RESULTS

Results of theory and experiment are compared in Fig. 5. On the abscissa we have introduced the parameter α, which is the factor by which the rf potential is flattened:

$$\alpha \equiv \frac{V_0}{V_{\text{eff}}} = \frac{\omega_{s0}^2}{\omega_s^2}. \tag{9}$$

Experimental points have been obtained for a maximum α of 1.73. α is the inverse of the "voltage reduction" k_t defined in Ref. [5]. The achieved values of α or k_t can be formally compared with the "local microwave instability threshold" (the Boussard criterion [13]), which is relevant if the high-frequency broadband impedance (responsible for an instability at wavelengths much shorter than the bunch length) is of the same magnitude as the space-charge impedance (causing the rf potential flattening). In Ref. [5] this threshold is calculated as $k_t = 0.6$ ($\alpha = 1.67$), which is equivalent to 40% of the space-charge limited current. We emphasize, however, that the prediction of instability according to the Boussard criterion in its original form is not applicable here, where the space-charge impedance is by far the dominant one.

Note that the experimental α values have been derived from the measured ω_s values according to Eq. (9), hence it is trivial that the measured points coincide with the analytical curve for ω_s. The independence of ω_1 from intensity is a clear identification of the dipole mode frequency.

The measured quadrupole frequencies are systematically lower than the analytical values and opposite for the sextupole mode. This is of a practical consequence if one uses the measured quadrupole frequency to calculate α. From the analytical expression in Eq. (5) we obtain

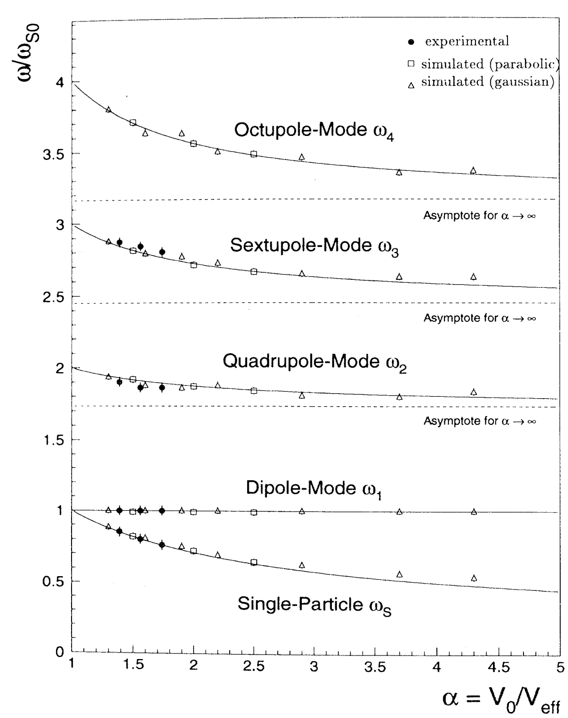

FIG. 5. Normalized frequencies vs rf potential reduction factor comparing analytical formulas (continuous lines), experiment, and simulation. Dashed lines are limiting values at the space-charge limit ($\alpha \to \infty$).

a nonlinear relationship

$$\frac{\omega_2^2}{\omega_{s0}^2} = 3 + \frac{1}{\alpha}. \tag{10}$$

This formula was recently used to calculate the effective potential for high-intensity cooled proton bunches in Ref. [3]. Values of α as large as 4 (i.e., $V_{\rm rf}/V_{\rm eff} \approx 0.25$ in Ref. [3]) have been suggested by this evaluation. By inspecting the quadrupole branch of Fig. 5 and using Eq. (10) we find, however, that we would have overestimated α by 50–70% for our experimental points. For an expected α as large as 4 the uncertainty is even larger due to the nonlinear dependence on α.

It should be noted here that an error in determining the actual $V_{\rm eff}$ leads to an error in determining $\Delta p/p$ for very space-charge-dominated bunches, since we readily obtain from Eq. (1) $\Delta p/p \propto z_0/\sqrt{\alpha}$. In order to explore possible discrepancies by computer simulation we have also compared in Fig. 5 the frequencies from a parabolic and a Gaussian bunch. The agreement with the analytical formula for the parabolic bunch justifies that the measurement of ω_2 allows an accurate calculation of the potential-well flattening factor α and thus $\Delta p/p$. The quadrupole frequencies of the simulated Gaussian bunch, however, show deviations of varying sign. We assume that the deviations are connected with the practical difficulty of obtaining a well-matched beam of a Gaussian distribution and possibly also with the nonlinearity of the space-charge force, which leads to an increase of the synchrotron frequency with amplitude. The simulation sextupole frequencies follow the tendency of the measured values and are slightly higher than the analytical values.

Hence it is concluded that for satisfactory accuracy in determining high phase-space density a direct measurement of the single-particle synchrotron frequency is necessary. As an alternative, the octupole frequencies are also promising, since they have a larger dispersion with α than the quadrupole frequencies or sextupole frequencies. If the spontaneous signal is insufficient (as was the case in our experiment), the mode can be excited by a higher harmonic rf bump. This is of practical consequence in heavy-ion-fusion storage rings, where values of α as large as 5–10, hence currents close to the space charge limit, are required.

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